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A family of degenerate Lie algebras ★

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Abstract

We show that almost all the real Lie algebras with only zero- and two-dimensional coadjoint orbits are degenerate in both the smooth and analytic category. The only exceptions are the already known cases (studied for example by Dufour and Weinstein).

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1. Introduction

A Poisson structure $\{\ ,\ \}$ on a manifold M is a Lie algebra structure on $C^{\infty}(M)$ satisfying the Leibniz identity:

$$\{fg, h\} + \{f, h\}, g + f\{g, h, \}, \quad \forall f, g, h \in C^{\infty}(M).$$

Alternatively it can be given by a contravariant skew-symmetric 2-tensor P such that $[P, P] = 0$, where $[\ ,\ ]$ stands for the Schouten bracket. In local coordinates the Poisson tensor P can be written in the form

$$P = \sum_{1 \leq i < j \leq n} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

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Using Weinstein's splitting theorem [6], the local study of P can be reduced to zero rank points, which translates into the following local expression for P:

$$P = \sum_{1\leq i<j\leq n} \sum_{k=1}^{n} C_{ij}^{k} x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \text{higher order terms.}$$

The numbers C_{ij}^{k} are the structure constants of a Lie algebra and they sometimes determine the possibility of bringing P to a (local) linear form

$$\sum_{1\leq i<j\leq n} \sum_{k=1}^{n} C_{ij}^{k} y_k \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j},$$

through a smooth or analytic change of coordinates. P is then said to be linearizable. In this context a Lie algebra is said to be (smoothly or analytically) nondegenerate if every Poisson tensor associated with it in the above way is (smoothly or analytically) linearizable. For example, semisimple Lie algebras are analytically nondegenerate (see [2,6]).

We classify, in terms of both the smooth and analytic nondegeneracy, all the Lie algebras of dimension higher than 3 whose connected and 1-connected Lie group has only zero- and two-dimensional coadjoint orbits (such Lie algebras will be called *nice* and an exhaustive list of them can be found in [1]). We conclude that, apart from the already known cases (those to which the results of Weinstein [6] and of Dufour [4] apply), all these Lie algebras are degenerate in both the smooth and analytic category. The proof is constructive, i.e., we associate a nonlinearizable Poisson structure to every Lie algebra $\mathfrak{g}$ being studied. This is done by perturbing the Lie–Poisson tensor in $\mathfrak{g}^*$ with second order terms in such a way that higher dimensional symplectic leaves appear around the singular point. This technique was used by Weinstein [7] to prove that noncompact semisimple Lie algebras of real rank at least 2 are smoothly degenerate.

Notation. We follow the notation in [1] for the nice Lie algebras. These are, up to a direct sum with a central ideal:

1. type (i) – $\mathfrak{so}(3)$ or $\mathfrak{sl}(2, \mathbb{R})$;
2. type (ii) – $\mathbb{R}T + \mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal and the action of T on $\mathfrak{a}$ is by an endomorphism of $\mathfrak{a}$;
3. type (iii) – $\mathbb{R}T + \mathfrak{h}$, where $\mathfrak{h}$ is the three-dimensional Heisenberg algebra spanned by X, Y, Z with $[X, Y] = Z$ and either

$$[T, X] = Y, \quad [T, Y] = -X, \quad [T, Z] = 0$$

   or

$$[T, X] = X, \quad [T, Y] = -Y, \quad [T, Z] = 0;$$

4. type (iv) – $\mathfrak{g}$ is six-dimensional with basis X_i, Y_i, $1 \leq i \leq 3$ and the nonvanishing brackets are

$$[X_1, X_2] = Y_3, \quad [X_2, X_3] = Y_1, \quad [X_3, X_1] = Y_2;$$

5. type(v) – $\mathfrak{g}$ is five-dimensional with basis $X_i, 1 \leq i \leq 3$, $Y_j, 1 \leq j \leq 2$ and the multiplicative law reads

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = Y_1, \quad [X_2, X_3] = Y_2.$$

The Lie–Poisson tensors in the dual of any of these Lie algebras will be denoted by *Lie–Poisson tensor of type (i)–(v).*

The main result is the following:

Theorem 1. *Let* $\mathfrak{g}$ *be a nice Lie algebra such that* $\dim \mathfrak{g} \geq 4$. *Then* $\mathfrak{g}$ *is smoothly and analytically degenerate except if* $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathbb{R}$ *or* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$.

We restrict ourselves to the case where the dimension of $\mathfrak{g}$ is at least 4, since the three-dimensional case was done by Dufour [3]. The proof of Theorem 1 can be found in Section 5.

2. Raising the rank of Poisson structures

Definition 1. A Lie algebra is said to be *nice* if the coadjoint orbits of its connected and 1-connected Lie group have dimension 0 or 2.

Definition 2. Let P be a Poisson tensor on a manifold M. Then P is said to be *nice* if its symplectic leaves have dimension 0 or 2 and *not nice at a singular point* if it has symplectic leaves of dimension at least 4, in some set whose closure contains the singular point.

We start with a nice Lie algebra $(\mathfrak{g}, [\ ,\ ])$, or equivalently with a nice Lie–Poisson tensor P on $V = \mathfrak{g}^*$. We want to perturb P with second order terms so that symplectic leaves of higher dimension appear in any neighbourhood of the origin. Let $(x_1, \ldots, x_n)$ (with $n \geq 4$) be linear coordinates on V and P be a linear Poisson tensor on V. Then the expression of P in the basis

$$\left\{ \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right\}_{1 \leq i < j \leq n}$$

is linear. Let Q be an alternating contravariant 2-tensor whose expression in the above basis is quadratic. Then $P' = P + Q$ is said to be a *quadratic perturbation of* P. Such P' will be a Poisson tensor if and only if

$$[P + Q, P + Q] = 0,$$

where $[\ ,\ ]$ stands for the Schouten bracket. Equivalently

$$[P, Q] = 0 \quad \text{and} \quad [Q, Q] = 0. \tag{1}$$

The last equation means that Q itself is a Poisson tensor. Our goal is then to find a quadratic Poisson tensor Q such that $[P, Q] = 0$ and $P + Q$ is not nice at the origin.

3. Lie–Poisson tensors of type (ii)

In this section we build nonlinearizable perturbations of the Lie–Poisson tensors of type (ii). This shows that Lie algebras of type (ii) are smoothly and analytically degenerate. We recall that a Lie algebra of type (ii) is the semidirect sum of $\mathbb{R}$ and an n-dimensional abelian ideal $\mathfrak{a}$, where the action of $\mathbb{R}$ on $\mathfrak{a}$ is by an endomorphism E of $\mathfrak{a}$. We will denote such Lie algebra by $\mathfrak{g}_E$. The Lie–Poisson tensor on $\mathfrak{g}_E^*$ will be denoted by L_E.

Let $\{t\}$ be a generator of $\mathbb{R}$ and $\{x_1, \ldots, x_n\}$ be any basis for $\mathfrak{a}$. Then the bracket relations in $\mathfrak{g}_E$ are

$$[t, x_i] = E(x_i) \quad \text{and} \quad [x_i, x_j] = 0.$$

Identifying the basis $\{t, x_1, \ldots, x_n\}$ for $\mathfrak{g}_E$ with a system of coordinates for its dual we can write the Lie–Poisson tensor L_E as

$$L_E = \sum_{i=1}^{n} E(x_i) \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x_i}. \tag{2}$$

Remark. *Dufour [3] has worked with this family of Lie algebras, which he divided into two subfamilies: the subfamily of nonresonant and that of resonant Lie algebras, which correspond, respectively, to the case where the eigenvalues of E are nonresonant and resonant. Dufour proved that resonant Lie algebras are smoothly degenerate and that three-dimensional nonresonant ones are smoothly nondegenerate. We will prove that in dimension higher than 3 all the Lie algebras* $\mathfrak{g}_E$ *are smoothly and analytically degenerate.*

Theorem 2. *Let* $E : \mathfrak{a} \longrightarrow \mathfrak{a}$ *be a nonzero endomorphism of* $\mathfrak{a}$ *and suppose that* $\dim \mathfrak{a} \geq 3$. *Then there exists a quadratic perturbation of* L_E *which is not nice at the origin.*

Proof. Following [5] we know that $\mathfrak{a}$ can be decomposed as a direct sum of invariant spaces (under E) of minimum dimensions. We call these subspaces *irreducible*. Furthermore, we know how E acts on each of these subspaces. We will consider a slightly different decomposition for $\mathfrak{a}$ which will make our proof easier:

$$\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k, \tag{3}$$

where $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ are irreducible subspaces of dimension greater than 1 and

$$\mathfrak{a}_0 = \mathfrak{a}_0^1 \oplus \cdots \oplus \mathfrak{a}_0^r,$$

where each $\mathfrak{a}_0^i$ is an invariant one-dimensional subspace (we remark that $\mathfrak{a}_0$ can be nonexistent). Then we know that there exists a basis for $\mathfrak{a}$ such that the action of E on $\mathfrak{a}_0$ is represented by the matrix

$$M_0 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix},$$

and the action of E on each of the subspaces $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ is represented by one of the two matrices

$$M = \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}$$

or

$$M' = \begin{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} & & & & \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} & & & \\ & & \cdots & & \\ & & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \end{pmatrix}$$

for convenient real numbers $\lambda_1, \ldots, \lambda_r, \lambda, \alpha$ and β, where $\beta \neq 0$ (blank entries correspond to null entries).

Now consider decomposition 3 for $\mathfrak{a}$. The map $E : \mathfrak{a} \longrightarrow \mathfrak{a}$ is just the product map (in the usual sense) of $E_0, E_1, \ldots, E_k$, where E_i stands for the restriction of E to $\mathfrak{a}_i$. The tensor L_E is the tensor which, in coordinates x^i for $\mathfrak{a}_i$, is written as

$$L_E(x^0, \ldots, x^k) = L_{E_0}(x^0) + \cdots + L_{E_k}(x^k).$$

We will prove the theorem in two steps. In the first step we produce perturbations for L_E in the case E is represented by one of the matrices M_0, M or M'. In the second step we show how to build a perturbation corresponding to the action of E on $\mathfrak{a}_i \oplus \mathfrak{a}_j$ by "adding" perturbations which correspond to the actions of E on $\mathfrak{a}_i$ and $\mathfrak{a}_j$. We then show that the perturbation built in this way is not nice. □

3.1. Perturbations of the tensors associated with M_0, M and M'

Lemma 1. *Let E be represented by the matrix M_0. Then we can find Q_E such that $L_E + Q_E$ is a quadratic perturbation of L_E. Furthermore the perturbation can be chosen to be not nice at the origin unless $r \leq 2$ or all the λ's are zero.*

Proof. Assume that $r \geq 2$ and let $(x_1, \ldots, x_r)$ be the basis for $\mathfrak{a}$ such that E is represented by M_0. This means that

$$L_E(x) = \sum_{i=1}^{r} \lambda_i x_i \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x_i}.$$

Let

$$Q_E = a(x) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2},$$

where a is some quadratic function. Then, Q_E is a Poisson tensor and the equation $[L_E, Q_E] = 0$ is equivalent to

$$(\lambda_1 + \lambda_2)a = \lambda_1 x_1 \frac{\partial a}{\partial x_1} + \cdots + \lambda_r x_r \frac{\partial a}{\partial x_r}.$$

This is a partial differential equation which is singular at the origin so that Cauchy–Kowalevsky's theorem does not apply. However, because we are interested in quadratic solutions of this equation we find easily a solution to be $a(x) = Ax_1x_2$, producing

$$Q_E = Ax_1x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}.$$

It is easy to check that $L_E + Q_E$ has four-dimensional symplectic leaves in any neighbourhood of the origin as long as one of $\lambda_3, \ldots, \lambda_r$ is nonzero. This can always be achieved by changing the order of the elements in the basis unless all λ_i's are zero or $r = 2$. In those cases we will be left with zero- and two-dimensional symplectic leaves.

In the case $r = 1$ we are forced to consider a different Q_E. We choose

$$Q_E = Ax^2 \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x}. \qquad \square$$

Lemma 2. *Let E be represented by the matrix M. Then there exists a quadratic perturbation $L_E + Q_E$ of L_E which is not nice if $r \geq 3$ and nice otherwise.*

Proof. As before let $(x_1, \ldots, x_r)$ be the basis where E is as in M, and let

$$Q_E = Ax_r^2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}.$$

Again Q_E is a Poisson tensor and the equation $[L_E, Q_E] = 0$ is satisfied. Furthermore, if $r \geq 3$, then $L_E + Q_E$ is not nice at the origin except in the case $\lambda = 0$. In this case we consider Q_E to be

$$(Ax_2x_3 + Bx_3^2) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + Ax_3^2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}.$$

The perturbed tensor $L_E + Q_E$ is indeed not nice as long as $A \neq 0$. $\square$

Lemma 3. *Let E be represented by M'. Then there is a quadratic perturbation $L_E + Q_E$ of L_E which, as long as $r \geq 3$, will raise its rank.*

Proof. It is easy to see that

$$Q_E = A(x_{r-1}^2 + x_r^2) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$$

produces the desired perturbation. This solution does raise the rank of L_E as long as $r \geq 3$. $\square$

3.2. Perturbations associated to a general endomorphism

The following lemma will guarantee that we can "add" perturbations in order to produce perturbations.

Lemma 4. *Let $(x', x_m, x'') = (x_1, \ldots, x_m, \ldots, x_n)$ be any system of coordinates in a manifold M and let P and Q be any two Poisson tensors on M satisfying*

$$P = \sum_{1 \le i < j \le m} P_{ij}(x') \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

and

$$Q = \sum_{m \le r < s \le n} Q_{rs}(x'') \frac{\partial}{\partial x_r} \wedge \frac{\partial}{\partial x_s}.$$

Then $[P, Q] = 0$.

Proof. Just computational. □

Lemma 5. *Let $E_i : \mathfrak{a}_i \longrightarrow \mathfrak{a}_i$ and $E_j : \mathfrak{a}_j \longrightarrow \mathfrak{a}_j$ be two endomorphisms of any of the types described in Lemmas 1–3 and consider the product map $E_i \times E_j : \mathfrak{a}_i \oplus \mathfrak{a}_j \longrightarrow \mathfrak{a}_i \oplus \mathfrak{a}_j$ of E_i and E_j. Then we can find a quadratic perturbation of the tensor associated with the map $E_i \times E_j$ whose quadratic terms do not depend on t. Furthermore, if* $\dim \mathfrak{a}_i + \dim \mathfrak{a}_j \ge 3$, *such a perturbation can be chosen to be not nice at the origin.*

Proof. Let $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_s)$ be bases for $\mathfrak{a}_i$ and $\mathfrak{a}_j$, respectively, as in Lemmas 1–3. Let Q_{E_i} and Q_{E_j} be as in those lemmas and let

$$P'(x, y) = L_{E_i}(x) + Q_{E_i}(x) + L_{E_j}(y) + Q_{E_j}(y).$$

We recall that

$$L(x, y) = L_{E_i}(x) + L_{E_j}(y)$$

is the tensor associated with the product map $E_i \times E_j$. We show first that P' is a Poisson tensor. Because L, $L_{E_i} + Q_{E_i}$ and $L_{E_j} + Q_{E_j}$ are Poisson tensors this amounts to showing that

$$[L_{E_i}, Q_{E_j}] = [L_{E_j}, Q_{E_i}] = [Q_{E_i}, Q_{E_j}] = 0.$$

Each of these Schouten brackets vanishes as a consequence of Lemma 4 (take $x_m = t$ in that lemma). Furthermore the perturbation P' just built for the tensor associated with $E = E_i \times E_j$ is again of the type

$$L_E(x, y) + Q_E(x, y)$$

and therefore Lemma 4 shows that the induction process can go on.

We now show that the dimension of the symplectic leaves is as stated. If $r \geq 3$ or $s \geq 3$ then one of the Lemmas 1–3 makes sure that this is the case. We are left with the cases $(r, s) = (1, 2)$ and $(r, s) = (2, 2)$ (the first producing two cases and the second producing three cases). The matrices which represent the product map in these five cases, together with the corresponding perturbations are as follows:

$$\begin{pmatrix} \lambda & & \\ & \mu & \\ & 1 & \mu \end{pmatrix}, \quad Q_E = Ax^2 \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x} + By_2^2 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2},$$

$$\begin{pmatrix} \lambda & & \\ & \alpha & -\beta \\ & \beta & \alpha \end{pmatrix}, \quad Q_E = Ax^2 \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x} + B(y_1^2 + y_2^2) \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2},$$

$$\begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & & \alpha & -\beta \\ & & \beta & \alpha \end{pmatrix}, \quad Q_E = Ax_2^2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + B(y_1^2 + y_2^2) \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2},$$

$$\begin{pmatrix} \lambda & & & \\ & \mu & & \\ & & \alpha & -\beta \\ & & \beta & \alpha \end{pmatrix}, \quad Q_E = Ax_1x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + B(y_1^2 + y_2^2) \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2},$$

$$\begin{pmatrix} \alpha & -\beta & & \\ \beta & \alpha & & \\ & & \gamma & -\delta \\ & & \delta & \gamma \end{pmatrix}, \quad Q_E = A(x_1^2 + x_2^2) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + B(y_1^2 + y_2^2) \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}.$$

It is easy to see that in all five cases the real numbers A and B can be chosen so that the perturbation is not nice. This concludes the proof of the lemma.

It also concludes the proof of the theorem, since from one step to the other the essential properties of the tensors involved are preserved. □

4. Nice tensors admitting Casimir functions

Let P be a nice Lie–Poisson tensor on a vector space V which admits a coordinate function (say x_n) as a Casimir function. This is the case of the tensors of type (iii)–(v). We will consider simpler perturbations $P + Q$ of P by taking Q to be of the form $x_n L$, with L a Lie–Poisson tensor.

Lemma 6. *Let P and L be nice Lie–Poisson tensors on V and x_n a Casimir function for P. Then $P + x_n L$ is a Poisson tensor if and only if $[P, L] = 0$.*

Proof. Using properties of the Schouten bracket we can write

$$[P + x_n L, P + x_n L] = [P, P] + 2[P, x_n L] + [x_n L, x_n L].$$

Now $[P, P] = 0$ since P is a Poisson tensor and

$$[x_n L, x_n L] = x_n^2 [L, L] - 2 x_n L^\sharp(\mathrm{d}x_n) \wedge L.$$

Again $[L, L] = 0$ and the fact that L is nice implies that $L^\sharp(\mathrm{d}x_n) \wedge L = 0$, so that

$$[P + x_n L, P + x_n L] = 2[P, x_n L] = 2x_n [P, L] - 2P^\sharp(\mathrm{d}x_n) \wedge L.$$

The conclusion follows using the fact that x_n is a Casimir function for P. □

We consider now the problem of raising the rank of P.

Lemma 7. *Let P and L be as in the previous lemma and suppose that*
1. $[P, L] = 0$;
2. *there is a subset U of V, whose closure contains the origin, where the following conditions hold:*
   (a) $\mathrm{im}\,(P^\sharp) \cap \mathrm{im}\,(L^\sharp) = \{0\}$;
   (b) $\ker(P^\sharp) \neq \ker(L^\sharp)$.

Then the tensor $P + x_n L$ is not nice at the origin.

Proof. First we remark that, if L is any nontrivial Lie–Poisson tensor, then the set

$$M_0(L) = \{p \in V\colon \mathrm{rank}(L)_p = 0\}$$

is a hyperplane of codimension at least 1. Since both P and L are in these conditions, and furthermore they are nice, then this implies that in any neighbourhood of the origin, there is a point p such that

$$\mathrm{rank}(P)_p = 2 \quad \text{and} \quad \mathrm{rank}(L)_p = 2.$$

Furthermore we can choose p such that $x_n(p) \neq 0$. The hypothesis on the image of P and L then implies that

$$\ker(P^\sharp + x_n L^\sharp)_p = \ker P_p^\sharp \cap \ker L_p^\sharp.$$

Since both $\ker P_p^\sharp$ and $\ker L_p^\sharp$ have codimension 2, the hypothesis on the kernel of $P^\sharp$ and $L^\sharp$ implies that $\ker(P^\sharp + x_n L^\sharp)_p$ has codimension 4, which concludes the lemma. □

Our goal is now to find L such that conditions 1, 2(a) and 2(b) of Lemma 7 hold.

4.1. Choice of L

We will choose L from the Lie–Poisson tensors of type (ii), as this will give us some freedom to choose (by choosing the endomorphism E). We will keep the notation of Section 3 for such L and denote it by L_E. We first write L_E in a coordinate free way. Let V be the vector space which is the base space for P. Then a Lie algebra $\mathfrak{g}_E$ of type (ii) on V^* is determined by $\alpha \in V$, $z \in V^*$ (with $z(\alpha) \neq 0$) and E a nonzero endomorphism of $\ker(\alpha)$, in the following sense:

$$\mathfrak{g}_E = \mathbb{R}z + \ker(\alpha), \tag{4}$$

where the action of z on $\ker(\alpha)$ is by the endomorphism E. Our objective is to show that there exist α, z and E such that the Lie–Poisson tensor L_E satisfies conditions 1, 2(a) and 2(b). Let $\beta = \{X_1, \dots, X_n\}$ be a basis for V and $(x_1, \dots, x_n)$ be coordinates in that basis. We write α as $\alpha_1 X_1 + \cdots + \alpha_n X_n$ and assume that $\alpha_1 \neq 0$. Assuming furthermore that $z(\alpha) = 1$, the expression of L_E in x-coordinates is given by

$$L_E(\mathrm{d}x_1, \mathrm{d}x_i) = E(\alpha_1 x_i - \alpha_i x_1)$$

and

$$L_E(\mathrm{d}x_i, \mathrm{d}x_j) = \frac{\alpha_i}{\alpha_1} E(\alpha_1 x_j - \alpha_j x_1) - \frac{\alpha_j}{\alpha_1} E(\alpha_1 x_i - \alpha_i x_1),$$

where $j > i > 1$. Now let u and v be two generators for the image of $P^\sharp$. Condition 2(a) is equivalent to saying that u and v form a free system together with the vector fields

$$u' = \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n \frac{\partial}{\partial x_n} \quad \text{and} \quad v' = E(x_1 - \alpha_1 z)\frac{\partial}{\partial x_1} + \cdots + E(x_n - \alpha_n z)\frac{\partial}{\partial x_n}$$

(see Lemma A.1 in Appendix A for the details).

We now remark that for fixed α and z it is easy to find an endomorphism E such that equation $[P, L] = 0$ holds. Conditions 2(a) and 2(b) merely restrict the field of those solutions. We choose $\alpha = X_1$ and $z = x_1$ so that $L_E(\mathrm{d}x_i, \mathrm{d}x_j)$ will be zero for all $j > i > 1$ and u' and v' will be given by

$$u' = \frac{\partial}{\partial x_1} \quad \text{and} \quad v' = E(x_2)\frac{\partial}{\partial x_2} + \cdots + E(x_n)\frac{\partial}{\partial x_n}.$$

Using these simplifications the problem of finding L_E is easily solved. The results can be found in Table 2.

In Table 1 we present (up to an isomorphism of coordinates in the base space V) the generators for the image and kernel of the nice Lie–Poisson tensors of types (ii)–(v). For the tensor of type (ii) we are using the just described choice of α and z and we denote by E_i the function $E(x_i)$. We have also assumed that $E_2 \neq 0$. This can always be achieved by permuting the coordinates $(x_2, \dots, x_n)$, unless $E = 0$. From this table it is easy to see that, taking L_E to be of type (ii), condition 2(b) holds automatically. The choice of E that forces L_E to satisfy conditions 1 and 2(a) can be found in Table 2. In that table we present the functions $E_2, \dots, E_n$ which determine the endomorphism E, where in (iv) one of E_4, E_5 or E_6 is nonzero and in (v) one of E_4 or E_5 is nonzero. This endomorphism, together with the just described choice of α and z, determines the Lie algebra $\mathfrak{g}_E$, and therefore the tensor L_E. We can therefore conclude that

Theorem 3. *Let P be one of the tensors P of type* (iii), (iv) *or* (v) *as denoted in* [1]. *Then there is a nice tensor L_E of type* (ii) *such that $P + x_n L_E$ is not nice at the origin.*

Table 1
Generators for the image and kernel of some nice Lie–Poisson tensors

Type of tensor	Generators for Image	Kernel
(ii)	$\frac{\partial}{\partial x_1}$, $E_2 \frac{\partial}{\partial x_2} + \cdots + E_n \frac{\partial}{\partial x_n}$	$E_2\, dx_i - E_i\, dx_2,\ i = 3, \ldots, n$
(iii)	$x_2 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2}$, $x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_3}$	dx_4, $x_4\, dx_1 + x_2\, dx_2 + x_3\, dx_3$
	$x_3 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_2}$, $x_2 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_3}$	dx_4, $x_4\, dx_1 + x_3\, dx_2 + x_2\, dx_3$
(iv)	$x_6 \frac{\partial}{\partial x_1} - x_4 \frac{\partial}{\partial x_3}$, $x_6 \frac{\partial}{\partial x_2} - x_5 \frac{\partial}{\partial x_3}$	dx_4, dx_5, dx_6, $x_4\, dx_1 + x_5\, dx_2 + x_6\, dx_3$
(v)	$x_4 \frac{\partial}{\partial x_1} + x_5 \frac{\partial}{\partial x_2}$, $x_3 \frac{\partial}{\partial x_1} - x_5 \frac{\partial}{\partial x_3}$	dx_4, dx_5, $x_5\, dx_1 - x_4\, dx_2 + x_3\, dx_3$

Table 2
Tensor L_E of type (ii) to be associated with P

Type of P	Endomorphism E determining L_E
(iii)	$E_2 = a_2x_2 + a_3x_3 + a_4x_4$ $E_3 = b_2x_2 + b_3x_3 + b_4x_4$ $E_4 = (a_2 + b_3)x_4 \neq 0$
(iv)	$E_2 = a_2x_2 + \cdots + a_6x_6$ $E_3 = b_2x_2 + \cdots + b_6x_6$ $E_4 = (a_2 + b_3)x_4$ $E_5 = c_4x_4 + c_5x_5 + c_6x_6$ $E_6 = d_4x_4 + d_5x_5 + d_6x_6$
(v)	$E_2 = a_2x_2 + \cdots + a_5x_5$ $E_3 = b_2x_2 + \cdots + b_5x_5$ $E_4 = c_4x_4 + c_5x_5$ $E_5 = (a_2 + b_3)x_5$

5. Proof of the main theorem

As proved in Theorems 2 and 3 the Lie algebras of types (ii)–(v) are degenerate in both the smooth and analytic category. In fact such theorems show that it is possible to perturb the Lie–Poisson tensor in the dual of these Lie algebras with second order terms in such a way that the perturbed tensor is no longer nice. This shows that such Lie algebras are degenerate in any category. Since abelian Lie algebras of dimension greater than 1 are degenerate, taking direct sums with central ideals will always produce a degenerate Lie algebra. This leaves us to classify

$$\mathfrak{so}(3) \oplus \mathbb{R} \quad \text{and} \quad \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}.$$

The Lie algebra $\mathfrak{so}(3) \oplus \mathbb{R}$ is smoothly and analytically nondegenerate as a consequence of the theorem in [4]. Using the same result one concludes that $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$ is analytically nondegenerate. It is, however, degenerate in the smooth sense since the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is smoothly degenerate (see [6]). This concludes the proof of Theorem 1. □

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Appendix A. Coordinate changes in the base space

Let $\phi : M \longrightarrow N$ be a diffeomorphism between manifolds M and N, and suppose that P is a Poisson tensor on M. Then there exists a unique Poisson tensor Q on N making ϕ into a Poisson diffeomorphism. Such tensor Q is given by $\phi_* P$, the pushforward of P by ϕ. Furthermore the image of $Q\sharp$ is just the pushforward by ϕ of the image of $P\sharp$. In the case we are interested in, ϕ is an automorphism of a vector space V. If ϕ is represented by the matrix A and the tensor matrix for P is M, then the tensor matrix for Q is just $N = AMA^{\mathrm{T}}$. Furthermore $\mathrm{im}(Q\sharp) = A(\mathrm{im}(P\sharp))$.

Lemma A.1. *Let V be a real vector space and let L_E denote the Lie–Poisson tensor on the dual of the Lie algebra*

$$\mathfrak{g}_E = \mathbb{R}z +_E \ker(\alpha).$$

Then there exist coordinates $x = (x_1, \dots, x_n)$ in V such that

$$L_E(\mathrm{d}x_1, \mathrm{d}x_j) = E(\alpha_1 x_j - \alpha_j x_1) \quad \text{for } j > 1$$

and

$$L_E(\mathrm{d}x_i, \mathrm{d}x_j) = \frac{\alpha_i}{\alpha_1} E(\alpha_1 x_j - \alpha_j x_1) - \frac{\alpha_j}{\alpha_1} E(\alpha_1 x_i - \alpha_i x_1) \quad \text{for } j > i > 1.$$

Furthermore the image of $L_E^\sharp$ is spanned by the vector fields

$$u = \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n \frac{\partial}{\partial x_n} \quad \text{and} \quad v = E(x_1 - \alpha_1 z)\frac{\partial}{\partial x_1} + \cdots + E(x_n - \alpha_n z)\frac{\partial}{\partial x_n}.$$

Proof. Let $(x_1, \dots, x_n)$ be coordinates in V such that $x_1(\alpha) \neq 0$. Then the following is a basis for $\ker(\alpha)$

$$\{y_2, \dots, y_n\} = \{\alpha_1 x_2 - \alpha_2 x_1, \dots, \alpha_1 x_n - \alpha_n x_1\},$$

where α_i stands for $x_i(\alpha)$. We complete this basis with $y_1 = z_1 x_1 + \cdots + z_n x_n$ to get a basis for $\mathfrak{g}_E$. In y-coordinates L is represented by the matrix

$$M = \begin{pmatrix} 0 & -E(y_2) & \cdots & -E(y_n) \\ E(y_2) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ E(y_n) & 0 & \cdots & 0 \end{pmatrix}.$$

The change of coordinates that takes us back to x coordinates is given by A the inverse matrix of

$$\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_n \\ -\alpha_2 & \alpha_1 & 0 & \cdots & 0 \\ -\alpha_3 & 0 & \alpha_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\alpha_n & 0 & 0 & \cdots & \alpha_1 \end{pmatrix}.$$

This is just

$$A = \begin{pmatrix} \alpha_1 & -z_2 & -z_3 & \cdots & -z_n \\ \alpha_2 & (1-\alpha_2 z_2)/\alpha_1 & -\alpha_2 z_3/\alpha_1 & \cdots & -\alpha_2 z_n/\alpha_1 \\ \alpha_3 & -\alpha_3 z_2/\alpha_1 & (1-\alpha_3 z_3)/\alpha_1 & \cdots & -\alpha_3 z_n/\alpha_1 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & -\alpha_n z_2/\alpha_1 & -\alpha_n z_3/\alpha_1 & \cdots & (1-\alpha_n z_n)/\alpha_1 \end{pmatrix}$$

and a tedious but straightforward calculation will complete the proof of the lemma. □

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